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Superalgebras for three interacting particles in an external magnetic field

J. Sadeghi[a]

Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5531, Tehran, Iran
and
Department of Physics Faculty of Sciences, Mazandaran University, P.O. Box 47415-416, Babolsar, Iran

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Abstract. In this paper we discuss interacting particles in an external magnetic field. By comparing the Schrödinger equation of three interacting particles with the associated Laguerre differential equation, we obtain the energy spectrum which corresponds to indices n_i and m_i. Finally by using the so called factorization method we obtain the raising and lowering operators. These operators are supersymmetric structures related to the Hamiltonian partner. Also these operators lead to the realization of Heisenberg Lie superalgebras with two, four and six supercharges.

PACS. 21.60.Cs Shell model

1 Introduction

In a recent paper [1], a new mechanism for the formation of three-particle clusters in the Si-MOSFET structures and GaAs /AlGaAS hetro junctions has been studied. These studies have shown that the exchange type interactions between $2D$ band electrons in the inversion layer and charged impurities in the oxide layer of the MOSFET can lead to an effective three-particle attractive interaction. An attraction between three particles leads to the formation of bound states with negative energies. However a weak attractive interaction between three particles does not produce a transition to a liquid state of the electron gas as described in Laughlin's approach [2,3]. The ground state with three-particle clustering could be energetically favored in the fractional quantum Hall regime.

We should stress that supersymmetry in quantum mechanics is based upon the factorization method in the framework of shape invariance. If a quantum mechanics problem allows supersymmetry, we must then factorize the Hamiltonian of quantum states in terms of a multiplication of the first-order differential operators as shape invariant equations. In this approach, the Hamiltonian is decomposed once in successive multiplication of lowering and raising operators. In such a way the corresponding quantum states of successive levels are the eigenstates of them. These Hamiltonians are called supersymmetric partners of each other. Initially the factorization method was suggested by Darboux [4] and later the application of this method was provided by Schrödinger in the framework of quantum mechanics [5]. To date, using the factorization method many studies on the one-dimensional shape invariance potential in the framework of supersymmetric quantum mechanics have been carried out [6–8]. Nowadays the concept of shape invariance has extended to ordinary differential equations and on this basis a second-order differential operator will decompose the multiplication of ladder operators [9].

In this paper, we use the factorization method and shape invariance of the associated Laguerre differential equation with respect to two parameters m_i and n_i and obtain the factorized Schrödinger equations for the three-interacting particles and also the energy spectrum. The supersymmetric structure corresponding to three-interacting particles describes the laddering relations for the parameters m_i and n_i respectively. These are interpreted as the Heisenberg Lie superalgebra $H \oplus H \oplus H$ for the three-interacting particles model. Also, we derive a nice symmetry involving simultaneous displacement of both parameters n_i and m_i for three-interacting particles quantum states. Finally, we can conclude the Heisenberg Lie superalgebra $H \oplus H \oplus H$ is realized by achieved operators.

2 Hamiltonian of the three-interacting particles

The Hamiltonian of the three-interacting particles in an external magnetic field can be constructed by,

$$H = \frac{-\hbar^2}{2M_e} \sum_{j=1}^{3} \left(\frac{\partial}{\partial \mathbf{r}_j} - i \frac{e}{\hbar c} \mathbf{A}_j \right) + \frac{k}{2} \Big[(\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_1 - \mathbf{r}_3)^2 + (\mathbf{r}_2 - \mathbf{r}_3)^2 \Big], \quad (1)$$

[a] e-mail: pouriya@ipm.ir

where $\mathbf{r}_j$ $(j = 1, 2, 3)$ are two- dimensional $(2D)$ position vectors with effective mass M_e.

The spring strength is chosen as $k = \frac{M_e \omega_0 2}{3}$, where ω_0^2 is the specific frequency of the electron's relative vibration. $\mathbf{A}_j = \frac{B}{2}\{-y, x, 0\}$ is the symmetric-gauge vector potential at the jth particle for a magnetic field $\mathbf{B} = \{0, 0, \frac{B}{2}\}$ which is perpendicular to the electronic inversion layer.

In order to solve the Hamiltonian (1), we diagonalize it by introducing the new coordinates $\{\mathbf{r}, \zeta, \eta\}$,

$$\begin{aligned} \mathbf{r} &= \frac{1}{\sqrt{3}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) \\ \zeta &= \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2) \\ \eta &= \sqrt{\frac{2}{3}}(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2} - \mathbf{r}_3), \end{aligned} \tag{2}$$

where $\mathbf{r}$ is the center-of-mass coordinate, ζ is the relative coordinate of the particle 1 and 2, η is the relative coordinate of the third particle with respect to the center — of — mass of the particle 1 and 2.

In this representation the Hamiltonian can be decomposed as,

$$H(\mathbf{r}, \zeta, \eta) = H(\mathbf{r}) + H(\zeta) + H(\eta), \tag{3}$$

where

$$H(\mathbf{r}) = \frac{-\hbar^2}{2M_e}\frac{\partial^2}{\partial \mathbf{r}^2} + \frac{3\hbar^2}{8M_e l_B^4}\mathbf{r}^2 + \frac{i\hbar^2}{2M_e {l_B}^2}(\mathbf{r} \times \nabla)_z, \tag{4}$$

$$H(\zeta) = \frac{-\hbar^2}{2M_e}\frac{\partial^2}{\partial \zeta^2} + \left(\frac{3\hbar^2}{8M_e l_B^4} + \frac{3k}{2}\right)\zeta^2 + \frac{i\hbar^2}{2M_e {l_B}^2}(\zeta \times \nabla)_z, \tag{5}$$

$$H(\eta) = \frac{-\hbar^2}{2M_e}\frac{\partial^2}{\partial \eta^2} + \left(\frac{3\hbar^2}{8M_e l_B^4} + \frac{3k}{2}\right)\eta^2 + \frac{i\hbar^2}{2M_e {l_B}^2}(\eta \times \nabla)_z, \tag{6}$$

where ${l_B}^2 = \frac{\hbar c}{eB}$ is the magnetic length.

The wave function of the initial Hamiltonian, equation (1) can be presented as the product of three wave functions of equations (4)–(6):

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \psi(r)\psi(\zeta)\psi(\eta).$$

Schrödinger's equations associated with the Hamiltonians given by equations (4)–(6) can be easily solved in a polar coordinate system,

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right)\Psi(\rho, \phi) + \frac{i}{{l_B}^2}\frac{\partial}{\partial \phi} + \frac{2M_e}{\hbar^2}\left(E - (\frac{\hbar^2}{8M_e {l_B}^4} + \frac{3k}{2})\rho^2\right)\Psi(\rho, \phi) = 0 \tag{7}$$

where ρ and ϕ are the radial and angular variables in the polar coordinate system.

The solution of this equation can be easily obtained by,

$$\Psi(\rho, \phi) = e^{im\phi}\Phi(\rho), \tag{8}$$

by defining K and ϵ,

$$\frac{\hbar^2}{8M_e {l_B}^4} + \frac{3k}{2} = K, \tag{9}$$

$$E - \frac{\hbar^2}{2M_e}\frac{m}{{l_B}^2} = \epsilon_m, \tag{10}$$

equation (7) becomes;

$$\Phi''(\rho) + \frac{1}{\rho}\Phi'(\rho) + \frac{2M_e}{\hbar^2}\left(\epsilon_m - \frac{\hbar^2}{2M_e}\frac{m^2}{\rho^2} - K\rho^2\right)\Phi(\rho) = 0. \tag{11}$$

With the definition of variable $z = A\rho^2$ and $\Phi(\rho) = R(z)L_{n,m}(z)$ we obtain the following Schrödinger equation,

$$zL''_{n,m}(z) + \left[1 + 2z\frac{R'}{R}\right]L'_{n,m}(z) + \left[z\frac{R''}{R} + \frac{R'}{R} + \frac{M_e}{2A\hbar^2}\left(\epsilon - \frac{\hbar^2 m^2}{2M_e}\frac{A}{z} - \frac{K}{A}z\right)\right]L_{n,m}(z) = 0. \tag{12}$$

3 Mathematical foundation

By supersymmetry approaches, we compute the parameters A, E and also the bound states $\Psi(\rho, \phi)$ from the comparison of the differential equation (12) with the associated Laguerre differential equation in an appropriate manner. Also here, we factorize the second order differential equations into the new sets of operators A^- and A^+ of shape invariant form, which are first order differential equations. This process is called the factorization method. To begin, we need to recall that for the real parameters $\alpha > -1$ and $\beta > 0$, the associated Laguerre differential equation corresponding to $L_{n,m}^{(\alpha,\beta)}(z)$ in the interval $z \in (0, \infty)$ is introduced as follows [10–12]:

$$zL''^{(\alpha,\beta)}_{n,m}(z) - [1 + \alpha - \beta z]\,L'^{(\alpha,\beta)}_{n,m}(z) + \left[(n - \frac{m}{2})\beta - \frac{m}{2}(\alpha + \frac{m}{2})\frac{1}{z}\right]L^{(\alpha,\beta)}_{n,m}(z) = 0. \tag{13}$$

Here, the indices n and m are non-negative integers with $0 \le m \le n$. The associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(z)$ as the solution of the differential equation (13) have the following Rodrigues representation,

$$L_{n,m}^{(\alpha,\beta)}(z) = \frac{a_{n,m}(\alpha,\beta)}{z^{\alpha+\frac{m}{2}}e^{-\beta z}}\left(\frac{d}{dx}\right)^{n-m}\left(z^{\alpha+n}e^{-\beta z}\right), \tag{14}$$

in which $a_{n,m}(z)$ is the normalization coefficient. As mentioned in references [10–12] and [13] we can write the associated Laguerre differential equation (13) as the following shape invariant equations with respect to the parameters m:

$$\begin{aligned} A_m^+(z)A_m^-(z)L_{n,m}^{(\alpha,\beta)}(z) &= (n - m + 1)\beta L_{n,m}^{(\alpha,\beta)}(z) \\ A_m^-(z)A_m^+(z)L_{n,m-1}^{(\alpha,\beta)}(z) &= (n - m + 1)\beta L_{n,m-1}^{(\alpha,\beta)}(z), \end{aligned} \tag{15}$$

where the explicit forms of operators $A_m^+(z)$ and $A_m^-(z)$ are respectively:

$$A_m^+(z) = \sqrt{z}\frac{d}{dz} - \frac{m-1}{2\sqrt{z}},$$
$$A_m^-(z) = -\sqrt{z}\frac{d}{dz} - \frac{2\alpha + m - 2\beta z}{2\sqrt{z}}. \quad (16)$$

One may write down the shape invariance equation (15) as the raising and lowering relations:

$$A_m^+(z)L_{n,m-1}^{(\alpha,\beta)}(z) = \sqrt{(n-m+1)\beta}L_{n,m}^{(\alpha,\beta)}(z),$$
$$A_m^-(z)L_{n,m}^{(\alpha,\beta)}(z) = \sqrt{(n-m+1)\beta}L_{n,m-1}^{(\alpha,\beta)}(z). \quad (17)$$

On the other hand, the associated Laguerre differential equation (13) can be factorized with respect to parameter n, for a given m as [10,11],

$$A_{n,m}^+(z)A_{n,m}^-(z)L_{n,m}^{(\alpha,\beta)}(z) = (n-m)(n+\alpha)L_{n,m}^{(\alpha,\beta)}(z),$$
$$A_m^-(z)A_{n,m}^+(z)L_{n-1,m}^{(\alpha,\beta)}(z) = (n-m)(n+\alpha)L_{n-1,m}^{(\alpha,\beta)}(z), \quad (18)$$

where the differential operators as functions of the parameters n and m are obtained as follows, respectively:

$$A_{n,m}^+(z) = z\frac{d}{dz} - \beta z + \frac{1}{2}(2n + 2\alpha - m),$$
$$A_m^-(z) = -z\frac{d}{dz} + \frac{1}{2}(2n - m). \quad (19)$$

Note that the shape invariance equations (18) can be written as the raising and lowering relations:

$$A_{n,m}^+(z)L_{n-1,m}^{(\alpha,\beta)}(z) = \sqrt{(n-m)(n+\beta)}L_{n,m}^{(\alpha,\beta)}(z),$$
$$A_{n,m}^-(z)L_{n,m}^{(\alpha,\beta)}(z) = \sqrt{(n-m)(n+\beta)}L_{n-1,m}^{(\alpha,\beta)}(z). \quad (20)$$

The method described, after some calculations, leads to the following normalization coefficient;

$$a_{n,m}(\alpha,\beta) = (-1)^m\sqrt{\frac{\beta^{\alpha+m+1}}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}}. \quad (21)$$

The normalization coefficient (21) has also been chosen so that the associated Laguerre functions $L_{n,m}^{\alpha,\beta}(z)$ with the same m but with different n with respect to the inner product with the weight functions $z^\alpha e^{-\beta z}$ forms an orthonormal set in the interval $0 \le z < \infty$:

$$\int_0^\infty L_{n,m}^{(\alpha,\beta)}(z)L_{n',m}^{(\alpha,\beta)}(z)z^\alpha e^{-\beta z}dz = \delta_{nn'}. \quad (22)$$

4 Interacting particles in a magnetic field

In order to obtain the energy spectrum, A and $\Psi(\rho,\phi)$ we need to compare equations (12) and (13). By using this formulation, we can obtain $R(z)$ and A,

$$R(z) = z^{\frac{\alpha}{2}}e^{-\frac{\beta}{2}z}, \quad (23)$$

$$A = \frac{1}{\beta\hbar}\sqrt{2M_eK} = \frac{1}{\beta}\sqrt{\frac{1}{4{l_B}^2} + \frac{2M_e}{\hbar^2}\frac{3}{2}k}, \quad (24)$$

where $k = \frac{M_e{\omega_0}^2}{3}$, $\omega_B = \frac{eB}{M_ec}$ (Cyclotron frequency). Finally one can rewrite A as follows,

$$A = \frac{1}{2\beta{l_B}^2}\sqrt{1 + \frac{4{\omega_0}^2}{{\omega_B}^2}}. \quad (25)$$

Also by comparing the equations (12) and (13), we obtain the energy spectrum as;

$$E_{n_i,m_i} = \hbar\omega_B\sqrt{1 + \frac{4{\omega_0}^2}{{\omega_B}^2}}\left[(n_i + \frac{\alpha}{2})\beta - \frac{(m_i-1)\beta}{2}\right] + \frac{m_i\hbar\omega_B}{2}. \quad (26)$$

If we compare the spectrum of the three-interacting particles with the spectrum of three free particles in a magnetic field we shall see that the former state has a lower energy. This statement shows that the additional term $-\frac{(m_i-1)\beta}{2}$ in equation (26) decreases the energy spectrum.

Here we also note that the index i in equation (26) describe the variables r, ζ and η, because they obey similar equations.

The wave function for three-interacting particles in an external magnetic field is,

$$\Psi(\rho(z)) = e^{im_i\phi}z^{\frac{\alpha}{2}}e^{-\frac{\beta}{2}z}L_{n_i,m_i}^{\alpha,\beta}(z). \quad (27)$$

Using the function $z^{\frac{\alpha}{2}}e^{-\frac{\beta}{2}z}$ on the operators $A_m^\pm(z)$ appearing in equations (15), together with application of the new variable ρ, $z = A\rho^2$ $(0 \le \rho < \infty)$ we can easily get the radial Schrödinger equation. According to this procedure, the explicit forms of the raising and lowering operators corresponding to the parameter m_i are calculated as,

$$A_{m_i}^\pm(\rho) = \pm\frac{1}{2\sqrt{A}}\frac{d}{d\rho} + W_{m_i}^\beta(\rho), \quad (28)$$

where $W_{m_i}^\beta(\rho)$ is the well-known three-interacting particles in an external field superpotential, which is,

$$W_{m_i}^\beta(\rho) = \frac{1}{2\sqrt{A}}\left[\frac{A\beta}{2}\rho - \frac{\alpha + m_i - 1}{2\rho}\right]. \quad (29)$$

Also, with the help of the associated Laguerre function $L_{n_i,m_i}^{(\alpha,\beta)}(z)$, we obtain the three-interacting particles quantum states corresponding to the supersymmetric partner [14],

$$A_{m_i}^+(\rho)A_{m_i}^-(\rho)\Psi_{n_i,m_i}(\rho) = (n_i - m_i + 1)\beta\Psi_{n_i,m_i}(\rho)$$
$$A_{m_i}^-(\rho)A_{m_i}^+(\rho)\Psi_{n_i,m_i-1}(\rho) = (n_i - m_i + 1)\beta\Psi_{n_i,m_i-1}(\rho), \quad (30)$$

as

$$\Psi_{n_i,m_i}(\rho) = A^{\frac{2\alpha+1}{4}}\rho^{\frac{2\alpha+1}{2}}e^{-\frac{A\beta}{2}\rho^2}L_{n_i,m_i}^{\alpha,\beta}(A\rho^2). \quad (31)$$

From equations (20), the raising and lowering relations for the quantum states of the three-interacting particles are as follows,

$$A^{+}_{m_i}(\rho)A^{-}_{m_i}(\rho)\Psi_{n_i,m_i-1}(\rho) = \sqrt{(n_i - m_i + 1)\beta}\Psi_{n_i,m_i}(\rho),$$
$$A^{-}_{m_i}(\rho)A^{+}_{m_i}(\rho)\Psi_{n_i,m_i}(\rho) = \sqrt{(n_i - m_i + 1)\beta}\Psi_{n_i,m_i-1}(\rho). \quad (32)$$

Now by using the orthonormal form of Laguerre polynomial and $\Psi_{n_i,m_i}(\rho)$ from equations (22) and (31), it is easily shown that the set of quantum states $\Psi_{n_i,m_i}(\rho)$ for a given value of m_i forms an orthonormal set,

$$\int_0^{\infty} \Psi_{n_i,m_i}(\rho)\Psi_{n'_i,m'_i}(\rho)d\rho = \delta_{n_i,n'_i}. \quad (33)$$

The operators $A^{+}_{m_i}(\rho)$ and $A^{-}_{m_i}(\rho)$ are hermitian conjugates of each other with respect to inner product (33). Shape invariance (30) describe radial part of the three — interacting particles in an external fields as supersymmetric Hamiltonian partners with the following partner potentials:

$$V^{(\pm;\beta)}_{m_i}(\rho) = W^2_{m_i}(\rho) \pm \frac{d}{2\sqrt{A}\rho} W_{m_i}(\rho), \quad (34)$$

so we have,

$$V^{(\pm;\beta)}_{m_i}(\rho) = \frac{A}{4}(\beta\rho^2) + \frac{1}{2A\rho^2}\left[\left(\alpha + m_i - 1\frac{\pm 1}{2}\right)(\alpha + m_i - 1)\right] - \frac{\beta}{2}\left(\alpha - 1 + m_i \mp \frac{1}{2}\right), \quad (35)$$

which satisfies the following shape invariance,

$$V^{+}_{m_i}(\rho) - V^{-}_{m_i+1}(\rho) = \text{ Const.}, \quad (36)$$

and constant is the function of parameter β which is just some number.

Also we know that the first — order differential equation (21) for $m_i = n_i + 1$ gives ground state $\Psi_{n_i,n_i}(\rho)$ as,

$$\Psi_{n_i,n_i}(\rho) = (-1)^{n_i}\sqrt{\frac{\beta^{n_i+\alpha+\frac{1}{2}}}{\Gamma(n_i + \alpha + \frac{1}{2})}} A^{\frac{n_i+\alpha}{2}} \rho^{n_i+\alpha} e^{-\frac{\beta A}{2}\rho^2}, \quad (37)$$

where it is in agreement with the analytic solution given for the ground state in (31). Now, using the equation (32) for given n_i, one can calculate algebraically all other quantum states by the ground states $\Psi_{n_i,n_i}(\rho)$,

$$\Psi_{n_i,m_i}(\rho) = \frac{A^{-}_{m_i+1}(\rho)A^{-}_{m_i+2}(\rho)...A^{-}_{n_i}(\rho)\Psi_{n_i,n_i}(\rho)}{\sqrt{\beta^{n_i-m_i}\Gamma(n_i + m_i + 1)}}, \quad (38)$$

where $m_i = 0, 1, 2, ..., n - 1$.

In order to obtain the raising and lowering operators of the first index n_i which describes radial quantization for the three-interacting quantum states, we do the similarity transformation by the function $z^{\frac{\alpha}{2}} e^{\frac{-\beta}{2}z}$ on the operators $A^{\pm}_{n_i,m_i}(z)$ of equation (20),

$$A^{\pm}_{n_i,m_i}(\rho) = \pm\frac{\rho}{2}\frac{d}{d\rho} - \frac{\beta}{2}A\rho^2 + \frac{1}{2}\left(2n_i + \alpha - 1 + m_i \mp \frac{1}{2}\right), \quad (39)$$

in which the radial variable ρ has been also used. Hence, the equation (20) yield the following relations for the raising and lowering of three-interacting particles quantum states with respect to first index n_i:

$$A^{+}_{n_i,m_i}(\rho)\Psi_{n_i-1,m_i}(\rho) = \sqrt{(n_i - m_i)(n_i + \alpha - \frac{1}{2})}\Psi_{n_i,m_i}(\rho),$$
$$A^{-}_{n_i,m_i}(\rho)\Psi_{n_i,m_i}(\rho) = \sqrt{(n_i - m_i)(n_i + \alpha - \frac{1}{2})}\Psi_{n_i-1,m_i}(\rho). \quad (40)$$

It is obvious that for a given m_i, one can derive the lowest state $\Psi_{m_i,m_i}(\rho)$ as equation (37) with m_i instead of n_i from the first-order differential equations (40) with $n_i = m_i$. So $\Psi_{n_i,m_i}(\rho)$ will be as

$$\Psi_{n_i,m_i}(\rho) = \sqrt{\frac{\Gamma(\alpha + \frac{1}{2} + m_i)}{\Gamma(\alpha + \frac{1}{2} + n_i)\Gamma(n_i - m_i + 1)}} \times A^{+}_{n_i,m_i}(\rho)A^{+}_{n_i-1,m_i}(\rho)...A^{+}_{n_i+1,m_i}(\rho)\Psi_{m_i,m_i}(\rho), \quad (41)$$

where $n_i = m_i + 1, m_i + 2, \cdots$

We now give an interesting feature of a simultaneous shape invariance with respect to both parameters n_i and m_i of the three-interacting particles quantum states $\Psi_{n_i,m_i}(\rho)$. Defining the differential operators of first -order as the following form which are hermitian conjugates of each other with respect to the inner product (33),

$$\begin{aligned} A^{+}_{m_i} &= A^{+}_{m_i}(\rho)A^{+}_{n_i,m_i-1}(\rho) - A^{+}_{n_i,m_i}(\rho)A^{+}_{m_i}(\rho) \\ &= \frac{1}{2\sqrt{A}}\frac{d}{d\rho} + W^{-\beta}_{m_i}(\rho), \\ A^{-}_{m_i} &= A^{+}_{n_i,m_i-1}(\rho)A^{+}_{m_i}(\rho) - A^{+}_{m_i}(\rho)A^{+}_{n_i,m_i}(\rho) \\ &= -\frac{1}{2\sqrt{A}}\frac{d}{d\rho} + W^{-\beta}_{m_i}(\rho), \end{aligned} \quad (42)$$

and with the help of equations (20),(22) and (40), one can obtain the following relations for the raising and lowering of the quantum states $\Psi_{n_i,m_i}(\rho)$ with respect to both parameters n_i and m_i,

$$A^{+}_{m_i}(\rho)\Psi_{n_i-1,m_i-1}(\rho) = \sqrt{(n + \alpha - \frac{1}{2})\beta}\Psi_{n_i,m_i}(\rho)$$
$$A^{-}_{m_i}(\rho)\Psi_{n_i,m_i}(\rho) = \sqrt{(n + \alpha - \frac{1}{2})\beta}\Psi_{n_i,m_i-1}(\rho). \quad (43)$$

As another interesting result, using the relations (43), one can obtain the factorized Schrödinger equations with respect to both parameters n_i and m_i,

$$A^{+}_{m_i}(\rho)A^{-}_{m_i}(\rho))\Psi_{n_i,m_i}(\rho) = \left(n + \alpha - \frac{1}{2}\right)\Psi_{n_i,m_i}(\rho),$$
$$A^{-}_{m_i}(\rho)A^{+}_{m_i}(\rho))\Psi_{n_i-1,m_i-1}(\rho) = \left(n + \alpha - \frac{1}{2}\right)\Psi_{n_i-1,m_i-1}(\rho), \quad (44)$$

which includes the three-interacting particles supersymmetric partner potentials $V_{m_i}^{(+,-\beta)}(\rho)$ and $V_{m_i}^{(-,-\beta)}(\rho)$, respectively.

The supersymmetric partner Schrödinger equations (44) are again the radial part of the Hamiltonian corresponding to the three-interacting particles. The energy levels are independent of m_i. Unlike equations (20) and (19), they are described only in terms of the radial quantum number n_i. Since the following relations are identically satisfied,

$$\begin{aligned} &A_{m_i}^{+}(\rho)A_{n_i,m_i-1}^{-}(\rho) - A_{n_i,m_i}^{-}(\rho)A_{m_i}^{+}(\rho) = 0, \\ &A_{n_i,m_i-1}^{+}(\rho)A_{m_i}^{-}(\rho) - A_{m_i}^{-}(\rho)A_{n_i,m_i}^{+}(\rho) = 0, \end{aligned} \quad (45)$$

it becomes obvious that we cannot obtain the different operators of first-order such that they increase one of the indices n_i and m_i and decrease the other index by one unit.

Therefore, we have obtained three different types of the laddering relations (32), (40) and (43) for the three-interacting particles quantum states. The existence of each relation leads us to the representation of the Heisenberg Lie superalgebra H. Meanwhile, using any type three (or different type two) of the mentioned laddering operators, we can construct the Heisenberg Lie superalgebra $H \oplus H \oplus H (H \oplus H)$. In order to do this, we need to define the supercharge $Q_j^{\pm}$ and the bosonic operator H_j, for $j = 1, 2, 3$ as 6×6 with the following matrix elements:

$$\begin{aligned} &(Q_1^+)_{jk} = \delta_{j1}\delta_{k6}A_{m_i}^{+} && (Q_1^-)_{jk} = \delta_{j6}\delta_{k1}A_{m_i}^{-}, \\ &(Q_2^+)_{jk} = \delta_{j2}\delta_{k5}A_{n_i,m_i}^{+} && (Q_2^-)_{jk} = \delta_{j5}\delta_{k2}A_{n_i,m_i}^{-}, \\ &(Q_3^+)_{jk} = \delta_{j3}\delta_{k4}A_{m_i}^{+} && (Q_3^-)_{jk} = \delta_{j4}\delta_{k3}A_{m_i}^{-}, \end{aligned} \quad (46)$$

and also we have,

$$\begin{aligned} &(H_1)_{jk} = \delta_{j1}\delta_{k1}A_{m_i}^{+}A_{m_i}^{-} + \delta_{j6}\delta_{k6}A_{m_i}^{-}A_{m_i}^{+}, \\ &(H_2)_{jk} = \delta_{j2}\delta_{k2}A_{n_i,m_i}^{+}A_{n_i,m_i}^{-} + \delta_{j5}\delta_{k5}A_{n_i,m_i}^{-}A_{n_i,m_i}^{+}, \\ &(H_3)_{jk} = \delta_{j3}\delta_{k3}A_{m_i}^{+}A_{m_i}^{-} + \delta_{j4}\delta_{k4}A_{m_i}^{-}A_{m_i}^{+}. \end{aligned} \quad (47)$$

One can conclude the (anti) commutation relations of the Heisenberg Lie algebra $H \oplus H \oplus H$ as follows,

$$\begin{aligned} &\{Q_j^+, Q_k^-\} = \delta_{jk}H_k, \\ &\{Q_j^+, Q_k^+\} = \{Q_j^-, Q_k^-\} = 0, \\ &\left[H_j, Q_k^{\pm}\right] = [H_j, H_k] = 0. \end{aligned} \quad (48)$$

Note that the Heisenberg Lie superalgebra $H_1 \oplus H_2$ can be extracted in a similar manner. We also recall that the supercharges $Q^{\pm}$ and the bosonic operator H are defined as,

$$Q^{\pm} = \sum_{j=1}^{3} Q_j^{\pm} \qquad H = \sum_{j=1}^{3} H_i, \quad (49)$$

which satisfy the (anti)commutation relations of Heisenberg Lie superalgebra H_2 ,

$$\begin{aligned} &\{Q^+, Q-\} = H, \\ &\{Q^+, Q^+\} = \{Q^-, Q^-\} = 0, \\ &\left[H, Q^{\pm}\right] = 0. \end{aligned} \quad (50)$$

In fact, the existence of every pair of raising and lowering operators in quantum mechanics via the introduction of two supercharges operators (Q^+, Q^-) and one bosonic operator (H) enable us to construct a representation of supersymmetry algebra.

5 Conclusion

In this paper we discuss three-interacting particles. We have obtained the energy spectrum and some parameters for the three-interacting particles problem system. Using the factorization method, we derived some laddering operators. We could conclude that these laddering operators lead to Heisenberg Lie superalgebra. This leads us to introduce some supercharge operators. The interesting problem here is to find a representation for the supersymmetry algebra using method of shape invariance for the three-interacting particles problem.

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